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FORCED OSCILLATIONS OF A CLASS OF PARABOLIC EQUATIONS WITH FUNCTIONAL ARGUMENTS

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1. Introduction

We are concerned with the parabolic equation with functional arguments

$$\begin{aligned}&\frac{\partial}{\partial t}\left(u(x,t)-\sum_{i=1}^{\ell}h_i(t)u(x,\rho_i(t))\right)\\&\quad-a(t)\Delta u(x,t)-\sum_{i=1}^{k}b_i(t)\Delta u(x,\tau_i(t))\\&\quad+c(x,t,(z_i[u](x,t))_{i=1}^{\tilde{m}})=f(x,t),\quad (x,t)\in\Omega\equiv G\times(0,\infty),\end{aligned}\tag{1}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. It is assumed that :

(H$_1$) $h_i(t)\in C^1([0,\infty);[0,\infty))$ $(i=1,2,...,\ell)$,
$b_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,2,...,k)$,
$a(t)\in C([0,\infty);[0,\infty))$ and $f(x,t)\in C(\overline{\Omega};[0,\infty))$;

(H$_2$) $\rho_i(t)\in C^1([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}\rho_i(t)=\infty$ $(i=1,2,...,\ell)$,
$\tau_i(t)\in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty}\tau_i(t)=\infty$ $(i=1,2,...,k)$;

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(H_3)

$$c(x,t,(\xi_i)_{i=1}^{\tilde{m}}) \in C(\overline{\Omega} \times \mathbb{R}^{\tilde{m}};\mathbb{R}),$$

$$c(x,t,(\xi_i)_{i=1}^{\tilde{m}}) \geq \sum_{i=1}^{m} p_i(t)\varphi_i(\xi_i) \text{ for } (x,t,(\xi_i)_{i=1}^{\tilde{m}}) \in \Omega \times [0,\infty)^{\tilde{m}},$$

$$c(x,t,(-\xi_i)_{i=1}^{\tilde{m}}) \leq -\sum_{i=1}^{m} p_i(t)\varphi_i(\xi_i) \text{ for } (x,t,(\xi_i)_{i=1}^{\tilde{m}}) \in \Omega \times [0,\infty)^{\tilde{m}},$$

where $[0,\infty)^j = [0,\infty) \times [0,\infty)^{j-1}$ $(j = 1,2,...,\tilde{m})$, $p_i(t) \in C([0,\infty);[0,\infty))$, $\varphi_i(\xi) \in C([0,\infty);[0,\infty))$, and $\varphi_i(\xi)$ are convex in $(0,\infty)$ $(i = 1,2,...,m)$;

(H_4)

$$z_i[u](x,t) = \begin{cases} u(x,\sigma_i(t)) & (i = 1,2,...,m), \\ \max_{s \in B_i(t)} u(x,s) & (i = m+1, m+2,...,m_1), \\ \displaystyle\sum_{j=1}^{N_i} \int_G K_{ij}(x,t,y)\omega_{ij}(u(y,\sigma_{ij}(t)))dy & \\ & (i = m_1+1, m_1+2,...,\tilde{m}), \end{cases}$$

where $\sigma_i(t) \in C([0,\infty);\mathbb{R})$ $(i = 1,2,...,m)$, $\lim_{t\to\infty} \sigma_i(t) = \infty$, $B_i(t)$ $(i = m+1, m+2,...,m_1)$ are closed bounded sets of $[0,\infty)$ such that $\lim_{t\to\infty} \min_{s\in B_i(t)} s = \infty$, $\sigma_{ij}(t) \in C([0,\infty);\mathbb{R})$ $(i = m_1+1, m_1+2,...,\tilde{m}; j = 1,2,...,N_i)$, $\lim_{t\to\infty} \sigma_{ij}(t) = \infty$, $K_{ij}(x,t,y) \in C(\overline{\Omega} \times \overline{G};[0,\infty))$, and $\omega_{ij}(s) \in C(\mathbb{R};\mathbb{R})$ are odd functions with the property that $\omega_{ij}(s) \geq 0$ for $s > 0$.

We consider two kinds of boundary conditions :

(B_1) $u = \psi$ on $\partial G \times (0,\infty)$,

(B_2) $\dfrac{\partial u}{\partial \nu} + \mu u = \tilde{\psi}$ on $\partial G \times (0,\infty)$,

where ψ, $\tilde{\psi} \in C(\partial G \times (0,\infty);\mathbb{R})$, $\mu \in C(\partial G \times (0,\infty);[0,\infty))$ and ν denotes the unit exterior normal vector to ∂G.

Definition 1. By a *solution* of equation (1) we mean a function $u(x,t) \in C^2(\overline{G} \times [t_{-1},\infty);\mathbb{R}) \cap C^1(\overline{G} \times [\hat{t}_{-1},\infty);\mathbb{R}) \cap C(\overline{G} \times$

$[\tilde{t}_{-1}, \infty); \mathbb{R})$ which satisfies (1), where

$$t_{-1} = \min\left\{0, \min_{1 \le i \le k}\left\{\inf_{t \ge 0} \tau_i(t)\right\}\right\},$$

$$\hat{t}_{-1} = \min\left\{0, \min_{1 \le i \le \ell}\left\{\inf_{t \ge 0} \rho_i(t)\right\}\right\},$$

$$\tilde{t}_{-1} = \min\left\{0, \min_{1 \le i \le m}\left\{\inf_{t \ge 0} \sigma_i(t)\right\}, \min_{\substack{m_1+1 \le i \le \tilde{m} \\ 1 \le j \le N_i}}\left\{\inf_{t \ge 0} \sigma_{ij}(t)\right\}\right\}.$$

Definition 2. A solution u of the boundary value problems (1), (B_i) ($i = 1, 2$) is said to be *oscillatory* in Ω if u has a zero in $G \times (t, \infty)$ for any $t > 0$.

In the case where $h_i(t)$ ($i = 1, 2, ..., \ell$) are nonpositive, the oscillation of (1) was studied by Tanaka and Yoshida [6] and equation (1) with $c(x, t, (z_i[u](x,t))_{i=1}^{\tilde{m}})$ replaced by $-c(x, t, (z_i[u](x,t))_{i=1}^{\tilde{m}})$ was investigated by Kusano and Yoshida [3].

The purpose of this paper is to obtain sufficient conditions for every solution of the boundary value problems (1), (B_i) ($i = 1, 2$) to be oscillatory in a cylindrical domain.

In Section 2 we reduce the multi-dimensional oscillation problems to one-dimensional problems for functional differential inequalities of neutral type by using the integral means of solutions. Sufficient conditions for the oscillation of functional differential inequalities are given in Section 3. Section 4 is devoted to the oscillation results for the boundary value problems (1), (B_i) ($i = 1, 2$).

2. Reduction to one-dimensional problems

In this section we reduce oscillation problems for (1) to oscillation problems for functional differential inequalities.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$-\Delta w = \lambda w \quad \text{in} \quad G,$$
$$w = 0 \quad \text{on} \quad \partial G$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G (see Courant and Hilbert [1]).

The following notation is used :

$$F(t) = \left(\int_G \Phi(x)dx\right)^{-1} \int_G f(x,t)\Phi(x)dx,$$
$$\Psi(t) = \left(\int_G \Phi(x)dx\right)^{-1} \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS,$$
$$\tilde{F}(t) = \frac{1}{|G|}\int_G f(x,t)dx,$$
$$\tilde{\Psi}(t) = \frac{1}{|G|}\int_{\partial G} \tilde{\psi}\, dS,$$

where $|G| = \int_G dx$. For any continuous functions $\theta(t)$ we use the notation :

$$[\theta(t)]_\pm = \max\{\pm\theta(t),\, 0\}.$$

Theorem 1. *Assume that the hypotheses* (H_1)–(H_4) *hold. If the functional differential inequalities*

$$\frac{d}{dt}\left(y(t) - \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t))\right) + \lambda_1 a(t)y(t)$$
$$+ \lambda_1 \sum_{i=1}^{k} b_i(t)y(\tau_i(t)) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \le \pm G(t) \qquad (2)$$

have no eventually positive solutions, then every solution u *of the boundary value problem* (1), (B_1) *is oscillatory in* Ω, *where*

$$G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(\tau_i(t)).$$

Proof. Suppose to the contrary that there exists a solution u of the problem (1), (B_1) which is nonoscillatory in Ω. First we assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. The hypothesis (H_3) implies that

$$c(x,t,(z_i[u](x,t))_{i=1}^{\tilde{m}}) \ge \sum_{i=1}^{m} p_i(t)\varphi_i(u(x,\sigma_i(t))) \quad \text{in} \quad G \times [t_1, \infty)$$

for some $t_1 \geq t_0$. Hence, from (1) we see that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(u(x,t) - \sum_{i=1}^{\ell} h_i(t)u(x,\rho_i(t))\right) \\
&\quad - a(t)\Delta u(x,t) - \sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t)) \\
&\quad + \sum_{i=1}^{m} p_i(t)\varphi_i(u(x,\sigma_i(t))) \leq f(x,t) \quad \text{in} \quad G \times [t_1,\infty).
\end{aligned} \tag{3}$$

Multiplying (3) by $\left(\int_G \Phi(x)dx\right)^{-1} \Phi(x)$ and then integrating over G yields

$$\begin{aligned}
&\frac{d}{dt}\left(U(t) - \sum_{i=1}^{\ell} h_i(t)U(\rho_i(t))\right) - a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx \\
&\quad - \sum_{i=1}^{k} b_i(t)K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx \\
&\quad + \sum_{i=1}^{m} p_i(t)K_\Phi \int_G \varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \\
&\leq F(t), \quad t \geq t_1,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
K_\Phi &= \left(\int_G \Phi(x)dx\right)^{-1}, \\
U(t) &= \left(\int_G \Phi(x)dx\right)^{-1} \int_G u(x,t)\Phi(x)dx.
\end{aligned}$$

It follows from Green's formula that

$$\begin{aligned}
&K_\Phi \int_G \Delta u(x,t)\Phi(x)dx \\
=&K_\Phi \int_{\partial G} \left[\frac{\partial u}{\partial \nu}(x,t)\Phi(x) - u(x,t)\frac{\partial \Phi}{\partial \nu}(x)\right] dS \\
&\quad + K_\Phi \int_G u(x,t)\Delta\Phi(x)dx \\
=& - K_\Phi \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS - \lambda_1 K_\Phi \int_G u(x,t)\Phi(x)dx \\
=& - \Psi(t) - \lambda_1 U(t), \quad t \geq t_1.
\end{aligned} \tag{5}$$

Analogously we have

$$K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx = -\Psi(\tau_i(t)) - \lambda_1 U(\tau_i(t)), \quad t \geq t_2 \tag{6}$$

for some $t_2 \geq t_1$. Applying Jensen's inequality [5, p.160], we obtain

$$K_\Phi \int_G \varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq \varphi_i\left(U(\sigma_i(t))\right), \quad t \geq t_2. \tag{7}$$

Combining (4)–(7) yields

$$\begin{aligned}
&\frac{d}{dt}\left(U(t) - \sum_{i=1}^{\ell} h_i(t)U(\rho_i(t))\right) + \lambda_1 a(t)U(t) \\
&\quad + \lambda_1 \sum_{i=1}^{k} b_i(t)U(\tau_i(t)) + \sum_{i=1}^{m} p_i(t)\varphi_i(U(\sigma_i(t))) \leq G(t)
\end{aligned}$$

for $t \geq t_2$. Moreover, it is clear that $U(t) > 0$ on $[t_2, \infty)$. This contradicts the hypothesis. If $u < 0$ in $G \times [t_0, \infty)$, it can be shown that

$$c(x,t,(z_i[u](x,t))_{i=1}^{\tilde{m}}) \leq -\sum_{i=1}^{m} p_i(t)\varphi_i(-u(x,\sigma_i(t))) \quad \text{in } G \times [t_1, \infty)$$

for some $t_1 \geq t_0$. Letting $v = -u$, we obtain

$$\frac{\partial}{\partial t}\left(v(x,t) - \sum_{i=1}^{\ell} h_i(t)v(x,\rho_i(t))\right)$$
$$- a(t)\Delta v(x,t) - \sum_{i=1}^{k} b_i(t)\Delta v(x,\tau_i(t))$$
$$+ \sum_{i=1}^{m} p_i(t)\varphi_i(v(x,\sigma_i(t))) \leq -f(x,t) \quad \text{in} \quad G \times [t_1,\infty).$$

Proceeding as in the case where $u > 0$, we are led to a contradiction. The proof is complete.

Theorem 2. *Assume that the hypotheses* (H_1)–(H_4) *are satisfied. If the functional differential inequalities*

$$\frac{d}{dt}\left(y(t) - \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t))\right) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \leq \pm\tilde{G}(t) \quad (8)$$

have no eventually positive solutions, then every solution u of the boundary value problem (1), (B_2) *is oscillatory in Ω, where*

$$\tilde{G}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(t)\tilde{\Psi}(\tau_i(t)).$$

Proof. Assume on the contrary, that there exists a solution u of the problem (1), (B_2) such that $u > 0$ in $G \times [t_0,\infty)$ for some $t_0 > 0$. Arguing as in the proof of Theorem 1, we observe that the inequality (3) holds for some $t_1 \geq t_0$. Dividing (3) by $|G|$ and then integrating over G yields

$$\frac{d}{dt}\left(\tilde{U}(t) - \sum_{i=1}^{\ell} h_i(t)\tilde{U}(\rho_i(t))\right) - a(t)\frac{1}{|G|}\int_G \Delta u(x,t)dx$$
$$-\sum_{i=1}^{k} b_i(t)\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx + \sum_{i=1}^{m} p_i(t)\frac{1}{|G|}\int_G \varphi_i(u(x,\sigma_i(t)))dx$$
$$\leq \tilde{F}(t), \quad t \geq t_1. \quad (9)$$

It follows from the divergence theorem that

$$
\begin{aligned}
\frac{1}{|G|}\int_G \Delta u(x,t)dx &= \frac{1}{|G|}\int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)dS \\
&= \frac{1}{|G|}\int_{\partial G} \left(-\mu \cdot u(x,t) + \tilde{\psi}\right) dS \\
&\le \tilde{\Psi}(t), \quad t \ge t_1.
\end{aligned} \tag{10}
$$

Analogously we obtain

$$
\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx \le \tilde{\Psi}(\tau_i(t)), \quad t \ge t_2 \tag{11}
$$

for some $t_2 \ge t_1$. An application of Jensen's inequality shows that

$$
\frac{1}{|G|}\int_G \varphi_i(u(x,\sigma_i(t)))dx \ge \varphi_i\left(\tilde{U}(\sigma_i(t))\right), \quad t \ge t_2. \tag{12}
$$

Combining (9)–(12) yields

$$
\frac{d}{dt}\left(\tilde{U}(t) - \sum_{i=1}^{\ell} h_i(t)\tilde{U}(\rho_i(t))\right) + \sum_{i=1}^{m} p_i(t)\varphi_i(\tilde{U}(\sigma_i(t))) \le \tilde{G}(t)
$$

for $t \ge t_2$, and furthermore $\tilde{U}(t)$ is positive on $[t_2, \infty)$. This contradicts the hypothesis. The case where $u < 0$ can be treated similarly, and we are led to a contradiction. The proof is complete.

3. First order functional differential inequalities

In this section we obtain sufficient conditions for the functional differential inequality

$$
\frac{d}{dt}\left(y(t) - \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t))\right) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \le q(t) \tag{13}
$$

to have no eventually positive solution, where $q(t) \in C([t_0,\infty);\mathbb{R})$ for some $t_0 > 0$.

Theorem 3. *Assume that :*

(i) $\sum_{i=1}^{\ell} h_i(t) \leq 1$ *and* $\rho_i(t) \leq t$ $(i = 1, 2, ..., \ell)$;

(ii) *there is an integer* $j \in \{1, 2, ..., m\}$ *such that* $\varphi_j(\xi)$ *is nondecreasing,* $\varphi_j(\xi) > 0$ *for* $\xi > 0$, $p_j(t) > 0$ *and*

$$\int_{t_0}^{\infty} p_j(t)dt = \infty \ ;$$

(iii) *there exists* $Q(t) \in C^1([t_0, \infty); \mathbb{R})$ *satisfying* $Q'(t) \geq q(t)$ *and* $\lim_{t\to\infty} Q(t) = 0$.

If the functional differential inequality

$$y'(t) + p_j(t)\varphi_j\left([y(\sigma_j(t)) + Q(\sigma_j(t))]_+\right) \leq 0 \tag{14}$$

has no eventually positive solution, then (13) *has no eventually positive solution.*

Proof. Suppose that $y(t)$ is an eventually positive solution of (13). There exists a number $t_1 \geq t_0$ for which $y(t) > 0$, $y(\rho_i(t)) > 0$ $(i = 1, 2, ..., \ell)$ and $y(\sigma_i(t)) > 0$ $(i = 1, 2, ..., m)$ for $t \geq t_1$. Letting

$$z(t) = y(t) - \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t)) - Q(t), \tag{15}$$

we see from (13) that

$$z'(t) \leq -p_j(t)\varphi_j(y(\sigma_j(t))) < 0, \quad t \geq t_1 \tag{16}$$

and hence $z(t)$ is decreasing for $t \geq t_1$. We claim that $\lim_{t\to\infty} z(t) = z_\infty > -\infty$. If $\lim_{t\to\infty} z(t) = -\infty$, then the hypothesis (iii) implies that $\lim_{t\to\infty} \tilde{z}(t) = -\infty$, where

$$\tilde{z}(t) = y(t) - \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t)). \tag{17}$$

In case $y(t)$ is bounded from above, we observe that $\tilde{z}(t)$ is bounded from below. Hence, $y(t)$ is not bounded from above. Then, there exists a number $t_2 \geq t_1$ such that $\tilde{z}(t_2) < 0$ and $\max_{t_1 \leq t \leq t_2} y(t) = y(t_2)$. Therefore, we obtain

$$\begin{aligned}
\tilde{z}(t_2) &= y(t_2) - \sum_{i=1}^{\ell} h_i(t_2) y(\rho_i(t_2)) \\
&\geq y(t_2) - \sum_{i=1}^{\ell} h_i(t_2) y(t_2) \\
&= \left(1 - \sum_{i=1}^{\ell} h_i(t_2)\right) y(t_2) \geq 0,
\end{aligned}$$

which contradicts the fact that $\tilde{z}(t_2) < 0$. Hence, it can be shown that $\lim_{t\to\infty} z(t) = z_\infty > -\infty$. Using (16), we see that

$$\begin{aligned}
0 &\leq \int_{t_1}^{t} p_j(s)\varphi_j(y(\sigma_j(s)))ds \\
&\leq \int_{t_1}^{t} \sum_{i=1}^{m} p_i(s)\varphi_i(y(\sigma_i(s)))ds \\
&\leq -\int_{t_1}^{t} z'(s)ds = z(t_1) - z(t) \leq z(t_1) - z_\infty
\end{aligned}$$

and therefore $p_j(t)\varphi_j(y(\sigma_j(t))) \in L^1(t_1, \infty)$. Since $y(t) > 0$ for $t \geq t_1$, we find that $\liminf_{t\to\infty} y(t) \equiv k_0 \geq 0$. If $k_0 > 0$, then

$$y(\sigma_j(t)) \geq \frac{k_0}{2}, \quad t \geq t_3$$

for some $t_3 \geq t_1$. We easily see that

$$\begin{aligned}
\int_{t_1}^{t} p_j(s)\varphi_j(y(\sigma_j(s)))ds &\geq \int_{t_3}^{t} p_j(s)\varphi_j(y(\sigma_j(s)))ds \\
&\geq \varphi_j\left(\frac{k_0}{2}\right) \int_{t_3}^{t} p_j(s)ds, \qquad (18)
\end{aligned}$$

which contradicts the fact that $p_j(t)\varphi_j(y(\sigma_j(t))) \in L^1(t_1, \infty)$ in view of the hypothesis (ii). Hence, we conclude that $k_0 = 0$, that is, $\liminf_{t\to\infty} y(t) = 0$. Since $\lim_{t\to\infty} Q(t) = 0$, we obtain $\lim_{t\to\infty} \tilde{z}(t) = z_\infty$. Taking the inferior limit as $t \to \infty$ of the inequality

$$y(t) = \tilde{z}(t) + \sum_{i=1}^{\ell} h_i(t) y(\rho_i(t)) \geq \tilde{z}(t),$$

we find that $z_\infty \leq 0$. Let $z_\infty < 0$. If $y(t)$ is not bounded from above, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ satisfying

$$\begin{aligned}
&\lim_{n\to\infty} t_n = \infty, \\
&\lim_{n\to\infty} y(t_n) = \infty, \\
&\max_{t_1 \leq t \leq t_n} y(t) = y(t_n).
\end{aligned}$$

Then we see that for all large n

$$\begin{aligned}
0 > \tilde{z}(t_n) &= y(t_n) - \sum_{i=1}^{\ell} h_i(t_n) y(\rho_i(t_n)) \\
&\geq \left(1 - \sum_{i=1}^{\ell} h_i(t_n)\right) y(t_n) \geq 0,
\end{aligned}$$

which yields a contradiction. Hence, we observe that $y(t)$ is bounded from above. Then, there exists $\limsup_{t\to\infty} y(t) \equiv \overline{y}_\infty$, and therefore there exists a sequence $\{\tilde{t}_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \tilde{t}_n = \infty$ and $\lim_{n\to\infty} y(\tilde{t}_n) = \overline{y}_\infty$. It is readily checked that

$$\begin{aligned}
\tilde{z}(\tilde{t}_n) &= y(\tilde{t}_n) - \sum_{i=1}^{\ell} h_i(\tilde{t}_n) y(\rho_i(\tilde{t}_n)) \\
&\geq y(\tilde{t}_n) - \overline{y}_\infty + \sum_{i=1}^{\ell} h_i(\tilde{t}_n)(\overline{y}_\infty - y(\rho_i(\tilde{t}_n))). \qquad (19)
\end{aligned}$$

Since

$$\liminf_{n\to\infty} (\overline{y}_\infty - y(\rho_i(\tilde{t}_n))) \geq \overline{y}_\infty - \limsup_{n\to\infty} y(\rho_i(\tilde{t}_n)) \geq 0,$$

taking inferior limit of (19) as $n \to \infty$ yields

$$0 > z_\infty \geq \liminf_{n\to\infty} y(\tilde{t}_n) - \overline{y}_\infty = 0.$$

This is a contradiction, and consequently we conclude that $z_\infty = 0$. We see that $z(t) > 0$ for $t \geq t_1$ in view of the fact that $z(t)$ is decreasing. Since $y(t) \geq z(t) + Q(t)$ and $y(t) > 0$ for $t \geq t_1$, we obtain

$$y(t) \geq [z(t) + Q(t)]_+, \quad t \geq t_1$$

and therefore

$$y(\sigma_j(t)) \geq [z(\sigma_j(t)) + Q(\sigma_j(t))]_+, \quad t \geq T$$

for some $T \geq t_1$. The hypothesis (ii) implies that

$$\varphi_j\left(y(\sigma_j(t))\right) \geq \varphi_j\left([z(\sigma_j(t)) + Q(\sigma_j(t))]_+\right), \quad t \geq T. \tag{20}$$

Combining (16) with (20) yields

$$z'(t) + p_j(t)\varphi_j\left([z(\sigma_j(t)) + Q(\sigma_j(t))]_+\right) \leq 0, \quad t \geq T$$

and consequently $z(t)$ is an eventually positive solution of (14). This contradicts the hypothesis and completes the proof.

Theorem 4. *Assume that the hypothesis* (ii) *of Theorem* 3 *holds. If*

$$\int_{t_0}^{\infty} p_j(t)\varphi_j\left([Q(\sigma_j(t))]_+\right) dt = \infty, \tag{21}$$

then (14) *has no eventually positive solution.*

Proof. Let $y(t)$ be an eventually positive solution of (14). Then, there exists a number $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\sigma_j(t)) > 0$ for $t \geq t_1$. Since

$$y(\sigma_j(t)) + Q(\sigma_j(t)) \geq Q(\sigma_j(t)), \quad t \geq t_1,$$

we obtain

$$[y(\sigma_j(t)) + Q(\sigma_j(t))]_+ \geq [Q(\sigma_j(t))]_+, \quad t \geq t_1$$

and therefore

$$\varphi_j\left([y(\sigma_j(t)) + Q(\sigma_j(t))]_+\right) \geq \varphi_j\left([Q(\sigma_j(t))]_+\right), \quad t \geq t_1.$$

Hence, (14) implies

$$y'(t) + p_j(t)\varphi_j\left([Q(\sigma_j(t))]_+\right) \leq 0, \quad t \geq t_1. \tag{22}$$

Integrating (22) over $[t_1, t]$ yields

$$\int_{t_1}^{t} p_j(s)\varphi_j\left([Q(\sigma_j(s))]_+\right) ds \leq -\int_{t_1}^{t} y'(s)ds \leq y(t_1),$$

which contradicts the hypothesis (21). The proof is complete.

Theorem 5. *Assume that the hypotheses* (i), (iii) *of Theorem* 3 *are satisfied. Assume, moreover, that :*

(i) *there exist positive constants* $\tilde{\rho}_i$ *such that*

$$\rho_i'(t) \geq \tilde{\rho}_i \quad (i = 1, 2, ..., \ell) \ ;$$

(ii) *there is an integer* $j \in \{1, 2, ..., m\}$ *for which*

$$p_j(t) \geq \tilde{p}_j \quad \textit{for some } \tilde{p}_j > 0,$$
$$0 \leq \sigma_j'(t) \leq \tilde{\sigma}_j \quad \textit{for some } \tilde{\sigma}_j > 0 \ ;$$

(iii) *for some* $\beta > 0$

$$\varphi_j(\xi) \geq \beta\,\xi \quad \textit{for } \xi > 0.$$

If the functional differential inequality

$$y'(t) + \beta\, p_j(t)y(\sigma_j(t)) \leq -\beta\, p_j(t)Q(\sigma_j(t)) \tag{23}$$

has no eventually positive solution, then (13) *has no eventually positive solution.*

Proof. Let $y(t)$ be an eventually positive solution of (13). Proceeding as in the proof of Theorem 3, the function $z(t)$ defined by (15) is decreasing on $[t_1, \infty)$ for some $t_1 \geq t_0$, and that there exists $\lim_{t\to\infty} z(t) = z_\infty > -\infty$. Since $\lim_{t\to\infty} Q(t) = 0$, we obtain

$$\lim_{t\to\infty} \tilde{z}(t) = z_\infty > -\infty, \tag{24}$$

where $\tilde{z}(t)$ is given by (17). The hypothesis (ii) and (13) imply that

$$\begin{aligned}\tilde{p}_j\beta \int_{t_1}^{t} |y(\sigma_j(s))|ds &\leq \int_{t_1}^{t} p_j(s)\varphi_j(y(\sigma_j(s)))ds \\ &\leq \int_{t_1}^{t} \sum_{i=1}^{m} p_i(s)\varphi_i(y(\sigma_i(s)))ds \\ &\leq -\int_{t_1}^{t} z'(s)ds = z(t_1) - z(t),\end{aligned}$$

and therefore $y(\sigma_j(t)) \in L^1(t_1, \infty)$. It is easy to see that

$$\int_{t_1}^{t} |y(\sigma_j(s))|ds \geq \int_{t_1}^{t} |y(\sigma_j(s))|\frac{\sigma_j'(s)}{\tilde{\sigma}_j}ds = \frac{1}{\tilde{\sigma}_j}\int_{\sigma_j(t_1)}^{\sigma_j(t)} |y(s)|ds.$$

Letting $t \to \infty$, we find that $y(t) \in L^1(t_2, \infty)$ for some $t_2 \geq t_1$. It follows from the hypothesis (i) that

$$\begin{aligned}\int_{t_2}^{t} \left|\sum_{i=1}^{\ell} h_i(s)y(\rho_i(s))\right| ds &\leq \sum_{i=1}^{\ell} \int_{t_2}^{t} |y(\rho_i(s))|\frac{\rho_i'(s)}{\tilde{\rho}_i}ds \\ &= \sum_{i=1}^{\ell} \frac{1}{\tilde{\rho}_i}\int_{\rho_i(t_2)}^{\rho_i(t)} |y(s)|ds\end{aligned}$$

and letting $t \to \infty$ yields

$$\sum_{i=1}^{\ell} h_i(t)y(\rho_i(t)) \in L^1(t_2, \infty).$$

Hence, we observe that $\tilde{z}(t) \in L^1(t_2, \infty)$, which, together with (24), implies $\lim_{t\to\infty} \tilde{z}(t) = 0$. Consequently, we conclude that

$\lim_{t\to\infty} z(t) = 0$, and therefore $z(t) > 0$ for $t \geq t_1$. Since $y(t) \geq z(t) + Q(t)$, the hypothesis (iii) and (13) imply that

$$\begin{aligned} 0 &\geq z'(t) + \beta p_j(t) y(\sigma_j(t)) \\ &\geq z'(t) + \beta p_j(t)(z(\sigma_j(t)) + Q(\sigma_j(t))), \quad t \geq t_1. \end{aligned}$$

Therefore, $z(t)$ is an eventually positive solution of (23). This contradicts the hypothesis and completes the proof.

4. Oscillations of functional parabolic equations

We can derive the oscillation results for the boundary value problems (1), (B_i) $(i = 1, 2)$ by combining the results of Sections 2 and 3.

Theorem 6. *Assume that the hypotheses* (H_1)–(H_4), (i), (ii) *of Theorem* 3 *are satisfied, and assume, moreover, that there exists* $\Theta(t) \in C^1((0,\infty);\mathbb{R})$ *such that* $\Theta'(t) = G(t)$ [*resp.* $\Theta'(t) = \tilde{G}(t)$], $\lim_{t\to\infty} \Theta(t) = 0$. *If the functional differential inequalities*

$$y'(t) + p_j(t)\varphi_j\left(\left[y(\sigma_j(t)) \pm \Theta(\sigma_j(t))\right]_+\right) \leq 0$$

have no eventually positive solutions, then every solution u of the boundary value problem (1), (B_1) [*resp.* (1), (B_2)] *is oscillatory in* Ω.

Proof. We note that Theorem 1 remains true if (2) is replaced by

$$\frac{d}{dt}\left(y(t) - \sum_{i=1}^{\ell} h_i(t) y(\rho_i(t))\right) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \leq \pm G(t).$$

The conclusion follows by combining Theorems 1–3.

Theorem 7. *Assume that the hypotheses* (H_1)–(H_4), (i), (ii) *of Theorem* 3 *are satisfied. Every solution u of the boundary value problem* (1), (B_1) [*resp.* (1), (B_2)] *is oscillatory in* Ω *if there exists* $\Theta(t) \in C^1((0,\infty);\mathbb{R})$ *such that* $\Theta'(t) = G(t)$ [*resp.* $\Theta'(t) = \tilde{G}(t)$], $\lim_{t\to\infty} \Theta(t) = 0$, *and*

$$\int_{t_0}^{\infty} p_j(t)\varphi_j\left([\Theta(\sigma_j(t))]_\pm\right) dt = \infty$$

for some $t_0 > 0$.

Proof. The conclusion follows from Theorems 4 and 6.

Theorem 8. *Assume that the hypotheses* (H_1)–(H_4), (i) *of Theorem* 3, *and* (i), (ii), (iii) *of Theorem* 5 *are satisfied, and assume, moreover, that* $\sigma_j(t) \le t$, $\sigma_j(t)$ *is nondecreasing in* $(0, \infty)$, *and there exists* $\Theta(t) \in C^1((0,\infty);\mathbb{R})$ *such that* $\Theta'(t) = G(t)$ [*resp.* $\Theta'(t) = \tilde{G}(t)$], $\lim_{t\to\infty} \Theta(t) = 0$. *If the functional differential inequalities*

$$y'(t) + \beta p_j(t) y(\sigma_j(t)) \le \mp \beta p_j(t) \Theta(\sigma_j(t))$$

have no eventually positive solutions, then every solution u *of the boundary value problem* (1), (B_1) [*resp.* (1), (B_2)] *is oscillatory in* Ω.

Proof. The conclusion follows by combining Theorems 1, 2 and 5.

Theorem 9. *Assume that the hypotheses* (H_1)–(H_4), (i) *of Theorem* 3, *and* (i), (ii), (iii) *of Theorem* 5 *are satisfied. Assume, moreover, that* $\sigma_j(t) \le t$ *and* $\sigma_j(t)$ *is nondecreasing in* $(0, \infty)$. *Every solution* u *of the boundary value problem* (1), (B_1) [*resp.* (1), (B_2)] *is oscillatory in* Ω *if there exists* $\Theta(t) \in C^1((0,\infty);\mathbb{R})$ *such that* $\Theta'(t) = G(t)$ [*resp.* $\Theta'(t) = \tilde{G}(t)$], $\lim_{t\to\infty} \Theta(t) = 0$, *and there is a sequence* $\{t_n\}_{n=1}^{\infty}$ *such that* :

$$\lim_{n\to\infty} t_n = \infty,$$

$$\int_{\sigma_j(t_n)}^{t_n} p_j(t)\,dt \ge \frac{1}{\beta},$$

$$\int_{\sigma_j(t_n)}^{t_n} p_j(t)\Theta(\sigma_j(t))dt + \beta \int_{\sigma_j(t_n)}^{t_n} p_j(t) \left(\int_{\sigma_j(t)}^{\sigma_j(t_n)} p_j(s)\Theta(\sigma_j(s))ds \right) dt = 0.$$

Proof. The conclusion follows from Theorem 8 and a result of Yoshida [7, Proposition 1].

Example 1. We consider the problem

$$\frac{\partial}{\partial t}\left(u(x,t)-\frac{1}{e^{2\pi}}\,u(x,t-2\pi)\right)-\frac{\partial^2 u}{\partial x^2}(x,t)-\frac{1}{e^{\pi}}\frac{\partial^2 u}{\partial x^2}(x,t-\pi)$$
$$+\frac{1}{e^{2\pi}}e^{t}\,u(x,t-2\pi)=(\sin x)\sin t,\quad (x,t)\in(0,\pi)\times(0,\infty), \tag{25}$$

$$u(0,t)=u(\pi,t)=0,\quad t>0. \tag{26}$$

Here $n=1$, $G=(0,\pi)$, $\Omega=(0,\pi)\times(0,\infty)$, $\ell=k=m=\tilde{m}=1$, $h_1(t)=e^{-2\pi}$, $\rho_1(t)=t-2\pi$, $a(t)=1$, $b_1(t)=e^{-\pi}$, $\tau_1(t)=t-\pi$, $p_1(t)=e^{-2\pi}e^{t}$, $\sigma_1(t)=t-2\pi$, $\varphi_1(\xi)=\xi$ and $f(x,t)=(\sin x)\sin t$. It is easy to see that $\lambda_1=1$, $\Phi(x)=\sin x$, $\Psi(t)=0$ and $F(t)=G(t)=(1/4)\pi\sin t$. Choosing $\Theta(t)=-(1/4)\pi\cos t$, we find that $\Theta(\sigma_1(t))=-(1/4)\pi\cos t$, and that

$$\int_{t_0}^{\infty}\frac{1}{e^{2\pi}}e^{t}\left[-\frac{1}{4}\pi\cos t\right]_{\pm}dt\geq\frac{\pi}{4e^{2\pi}}\int_{t_0}^{\infty}[-\cos t]_{\pm}dt=\infty.$$

Hence, it follows from Theorem 7 that every solution u of the problem (25), (26) is oscillatory in $(0,\pi)\times(0,\infty)$. One such solution is $u(x,t)=(\sin x)e^{-t}\sin t$.

Example 2. We consider the problem

$$\frac{\partial}{\partial t}\left(u(x,t)-\frac{1}{e^{\pi}}\,u(x,t-\pi)\right)-2\frac{\partial^2 u}{\partial x^2}(x,t)-e^{\pi}\frac{\partial^2 u}{\partial x^2}(x,t-\pi)$$
$$+u(x,t-2\pi)=-2(\cos x)e^{-t}\sin t,\quad (x,t)\in(0,\pi)\times(0,\infty), \tag{27}$$

$$-\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t)=0,\quad t>0. \tag{28}$$

Here $n=1$, $G=(0,\pi)$, $\Omega=(0,\pi)\times(0,\infty)$, $\ell=k=m=\tilde{m}=1$, $h_1(t)=e^{-\pi}$, $\rho_1(t)=t-\pi$, $a(t)=2, b_1(t)=e^{\pi}$, $\tau_1(t)=t-\pi$, $p_1(t)=1, \sigma_1(t)=t-2\pi$, $\varphi_1(\xi)=\xi$ and $f(x,t)=-2(\cos x)e^{-t}\sin t$. It is easily checked that $\tilde{\Psi}(t)=0$ and $\tilde{F}(t)=\tilde{G}(t)=0$. We can choose $\Theta(t)=0$. Since

$$\int_{\sigma_1(t_n)}^{t_n}p_1(t)dt=\int_{t_n-2\pi}^{t_n}dt=2\pi\geq 1$$

for any sequence $\{t_n\}$, Theorem 9 implies that every solution u of the problem (27), (28) is oscillatory in $(0,\pi)\times(0,\infty)$. In fact, $u(x,t)=(\cos x)e^{-t}\cos t$ is such a solution.

Remark 1. Theorem 5 holds true if the functional differential inequality (23) is replaced by

$$y'(t) + \beta \tilde{p}_j y(\sigma_j(t)) \le -\beta \tilde{p}_j Q(\sigma_j(t)).$$

Remark 2. Theorem 9 was established by Yoshida [7].

Remark 3. The case where $m = \tilde{m}$, $\psi = \tilde{\psi} = 0$, $f(x,t) = 0$, $\rho_i(t) = t - \rho_i$ $(\rho_i > 0)$ $(i = 1, 2, ..., \ell)$, $\tau_i(t) = t - \tau_i$ $(\tau_i > 0)$ $(i = 1, 2, ..., k)$, $\sigma_i(t) = t - \sigma_i$ $(\sigma_i > 0)$ $(i = 1, 2, ..., m)$, $\varphi_i(\xi) = \xi$ $(i = 1, 2, ..., m)$ was studied by Mishev and Bainov [4]. When $m = \tilde{m}$, $\psi = \tilde{\psi} = 0$, $f(x,t) = 0$, $\rho_i(t) = t - \rho_i$ $(\rho_i > 0)$ $(i = 1, 2, ..., \ell)$ and the hypothesis (iii) of Theorem 5 holds, the boundary value problems (1), (B_i) $(i = 1, 2)$ were treated by Cui [2].

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